1 The arc length of the Archimedean spiral

The Archimedean spiral is given by the formula $r = a + b\theta$ in polar coordinates, or in Cartesian coordinates:

$$\begin{aligned} x(\theta) &= (a + b\theta)\cos\theta, \\ y(\theta) &= (a + b\theta)\sin\theta \end{aligned}$$

The arc length of any curve is given by

$$s(\theta) = \int \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta$$

where $x'(\theta)$ denotes the derivative of x with respect to θ.
In our concrete case, it is

$$\begin{aligned} s(\Theta) &= \int \sqrt{(b\cos\theta - (a+b\theta)\sin\theta)^2 + (b\sin\theta + (a+b\theta)\cos\theta)^2} d\theta \\ &= \frac{-a^2\sinh^{-1}\left(\frac{2b^2\theta+2ab}{\sqrt{4b^2(b^2+a^2)-4a^2b^2}}\right)}{2\sqrt{b^2}} + \frac{\left(b^2+a^2\right)\sinh^{-1}\left(\frac{2b^2\theta+2ab}{\sqrt{4b^2(b^2+a^2)-4a^2b^2}}\right)}{2\sqrt{b^2}} + \\ &\quad + \frac{\theta\sqrt{b^2\theta^2+2ab\theta+b^2+a^2}}{2} + \frac{a\sqrt{b^2\theta^2+2ab\theta+b^2+a^2}}{2b} \end{aligned}$$

What we are actually after, though, is to find out the friggin' θs which have a certain distance d between them. So we actually need the inverse. Sage[1] to the rescue (as for the integral, by the way):

$$\theta(s) = \frac{-b^3\sinh^{-1}\left(\frac{b^2\theta+ab}{\sqrt{b^4}}\right) - a\sqrt{b^2}\sqrt{b^2\theta^2+2ab\theta+b^2+a^2} + 2b\sqrt{b^2}s}{b\sqrt{b^2}\sqrt{b^2\theta^2+2ab\theta+b^2+a^2}}$$

Now, this is all correct and nice – except for the obvious reductions that Sage did not perform yet – but pretty unusable for your purposes, I guess, as there is a θ on the right side, too. So is there something else?

Yes: there is: the Clackson scroll formula[2]. Basically, it gives you a rough estimate how much material you will need if you make a spiral with n turns and spacing D between:

$$s = \pi D n^2$$

The n and D are our θ and b in disguise:

[1] http://www.sagenb.org/
[2] http://en.wikipedia.org/wiki/Clackson_scroll_formula

$$
\begin{aligned}
n &= \frac{\theta}{2\pi} \\
D &= \frac{b}{2\pi}
\end{aligned}
$$

Unfortunately, a is assumed to be 0 in that formula. However, just shift your θ so that it does not start at 0, but at $\frac{a}{b}$ instead, and voila, the radius r starts at a again. (By the way, the same reasoning would lead to a formula $\theta(s) = \frac{-b\sinh^{-1}\theta+2s}{b\sqrt{\theta^2+1}}$, which *still* has θ on the right hand side.)

This is the rule-of-thumb solution, therefore:

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\begin{aligned}
s &= \pi\frac{b}{2\pi}\frac{\theta^2}{4\pi^2} = \frac{b\theta^2}{8\pi^2} \\
\theta &= 2\pi\sqrt{\frac{2s}{b}}
\end{aligned}
$$

Okay, what does it mean? It means that you should evaluate the Archimedean spiral not at multiples of a constant, but at the square root thereof.

Intuitively, this is so obvious as to defeat explanation (after having done all the calculation, though): as the circumference of the circles approximated by the spiral is quadratic in the distance to the center, of course, we have to take the square root of the length to approximate the distance of the spiral end point to the center, too.